

Soliton surfaces and generalized symmetries of integrable systems

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Abstract

This communication presents a detailed description of the symmetries of integrable systems which are used to construct the Fokas-Gel'fand formula for immersion of 2D-soliton surfaces, associated with such systems, into Lie algebras. In addition, it contains an exposition of the main tool used to study symmetries of these systems, which allows us to find the explicit integrated form of the surfaces. We determine that the sufficient condition for the applicability of the Fokas-Gel'fand immersion formula of a 2D-surface is that the vector field be a common symmetry of an integrable system and its linear spectral problem.

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The objective of this communication is to determine a new sufficiency condition for the integrated form of soliton surfaces through Lie algebras of symmetries of integrable systems. The identification of necessary (already formulated in [1, 2]) and sufficient conditions for the existence of 2D-surfaces in terms of invariance conditions for generalized symmetries allows the integration of the immersion functions explicitly. These surfaces include those

already known, namely those constructed from a conformal symmetry in the spectral parameter $\lambda \in \mathbb{C}$ (known as the Sym-Tafel formula for immersion [3, 4]), gauge symmetry [5, 6] and generalized symmetries of the integrable system initiated in (see the main theorem [1] and [2]) and further developed in [7, 8, 9]. This approach provides us with a bijective correspondence between symmetries of the integrable equations and surfaces F immersed into Lie algebras g . This link has been discussed in [1] and the immersion function is governed by the formula (given up to an additive g -valued constant $c(\lambda)$)

$$F([u], \lambda) = \Phi^{-1} \left(\alpha(\lambda) \frac{\partial \Phi}{\partial \lambda} + S([\theta], \lambda) \Phi + \frac{\partial \Phi}{\partial \theta^n} R_n \right) \in g \quad (1)$$

with tangent vectors defined by

$$D_{x^\alpha} F([\theta], \lambda) = \Phi^{-1} A^\alpha([\theta], \lambda) \Phi, \quad \alpha = 1, 2, \quad (2)$$

$$A^\alpha([\theta], \lambda) = \alpha(\lambda) \frac{\partial U^\alpha}{\partial \lambda} + \left(\frac{\partial S}{\partial x^\alpha} + [S, U^\alpha] \right) + \sum_{n=1}^N \left(\frac{D U^\alpha}{D \theta^n} \right) R_n \in g, \quad (3)$$

where the total derivatives are given by

$$D_{x^\alpha} = \frac{\partial}{\partial x^\alpha} + \theta_{J,\alpha}^k \frac{\partial}{\partial \theta_j^k}, \quad \alpha = 1, 2 \quad J = (j_1, \dots, j_n), \quad j_\alpha = 1, 2, \quad |J| = n.$$

Here $\alpha(\lambda)$ is an arbitrary function of a spectral parameter λ and the matrix $S([\theta], \lambda)$ is a g -valued function on the jet space N defined by $[\theta] = (x^1, x^2, \theta^k, \theta_j^k)$ and depends also on some spectral parameter λ . The set of scalar functions $\theta = (\theta^1, \dots, \theta^N)$ satisfies an integrable system of PDEs $\Delta[\theta] = 0$ in two independent variables x^1 and x^2 . The group G -valued function $\Phi([\theta], \lambda)$, corresponding to the Lie algebra g , satisfies the associated linear spectral problem (LSP)

$$\Lambda([\theta], \lambda) \equiv D_{x^\alpha} \Phi([\theta], \lambda) - U^\alpha([\theta], \lambda) \Phi([\theta], \lambda) = 0, \quad \alpha = 1, 2, \quad (4)$$

where the g -valued functions U^α are defined by the zero-curvature condition (ZCC)

$$\Delta[\theta] \equiv D_{x^2} U^1 - D_{x^1} U^2 + [U^1, U^2] = 0, \quad (5)$$

which is equivalent to the integrable system $\Delta[\theta] = 0$ and is required to be independent of the spectral parameter $\lambda \in \mathbb{C}$. The scalar functions $R = (R_1, \dots, R_n)$ are symmetries of the equations $\Delta[\theta] = 0$. The Fréchet derivative of Φ with respect to θ^n in the direction of R_n is denoted by $(D\Phi/D\theta^n)R_n$. Here we have adopted the summation convention over the repeated lower and upper indices unless otherwise stated. All functions, tensor fields and manifolds are assumed to be smooth. For uniformity of this presentation, we use the standard notation of the apparatus of vector fields and their prolongation in accordance with the book by P.J. Olver [10]. When dealing with the formula for the immersion function of 2D-surface in Lie algebras we use the prolongation formalism of vector fields instead of the notion of Fréchet derivatives. According to [7, 8, 9], an integrated form of a surface associated with a generalized symmetry (written in evolutionary form) of the ZCC $\Delta[\theta] = 0$,

$$w_R = R^k [\theta] \frac{\partial}{\partial \theta^k}, \quad (6)$$

and its first prolongation

$$\text{pr } w_R = w_R + (D_J R^k [\theta]) \frac{\partial}{\partial \theta_J^k}, \quad (7)$$

for which the following infinitesimal criteria for symmetry

$$\text{pr } w_R (D_{x^2} U^1 - D_{x^1} U^2 + [U^1, U^2]) = 0, \quad \text{whenever } \Delta[\theta] = 0 \quad (8)$$

holds, was proposed in a form equivalent to

$$F = \Phi^{-1} \frac{D\Phi}{D\theta^n} R_n = \Phi^{-1} (\text{pr } w_R \Phi). \quad (9)$$

This is so since the Fréchet derivative of Φ with respect to θ^n can be written in an equivalent form through the prolongation of the vector field w_R (see Proposition 5.25 of [10] p307). The g -valued function F determines an immersion of a 2D-surface in the Lie Algebra g whenever the tangent vectors (2) with matrices $A^\alpha = \text{pr } w_R U^\alpha$, $\alpha = 1, 2$, are linearly independent. We first show that the integrated form of surfaces given by (9) holds if and only if the generalized symmetry w_R is a common symmetry of both systems, namely of the ZCC $\Delta[\theta] = 0$ and its corresponding LSP $\Lambda([\theta], \lambda)$.

Indeed differentiating (9) and using the LSP (4), we obtain

$$D_{x^\alpha} F = D_{x^\alpha} \left(\Phi^{-1} \frac{D\Phi}{D\theta^n} R_n \right) = \Phi^{-1} \left[-U^\alpha \frac{D\Phi}{D\theta^n} R_n + D_{x^\alpha} \left(\frac{D\Phi}{D\theta^n} R_n \right) \right]. \quad (10)$$

Using Lemma 5.12 of ([10] p 300) the second term in (10) becomes

$$D_{x^\alpha} \left(\frac{D\Phi}{D\theta^n} R_n \right) = D_{x^\alpha} (\text{pr } w_R \Phi) = \text{pr } w_R (D_{x^\alpha} \Phi), \quad (11)$$

since the total derivatives D_α commute with the prolongation of the vector fields in evolutionary form, *i.e.* $[D_{x^\alpha}, \text{pr } w_R] = 0$. Using the identity

$$\text{pr } w_R (D_{x^\alpha} \Phi) = \text{pr } w_R (U^\alpha \Phi) + \text{pr } w_R (D_{x^\alpha} \Phi - U^\alpha \Phi), \quad (12)$$

we determine that the second term in (12) is not always zero. This term vanishes when the vector field w_R is also a symmetry of the LSP $\Lambda([\Phi], \lambda) = 0$. Therefore from (12) we obtain

$$\begin{aligned} \text{pr } w_R (U^\alpha \Phi) &= (\text{pr } w_R U^\alpha) \Phi + U^\alpha (\text{pr } w_R \Phi) \\ &= \left(\frac{DU^\alpha}{D\theta^n} R_n \right) \Phi + U^\alpha \left(\frac{D\Phi}{D\theta^n} R_n \right). \end{aligned} \quad (13)$$

Hence from (10) we obtain

$$D_{x^\alpha} F = \Phi^{-1} \left[-U^\alpha \left(\frac{D\Phi}{D\theta^n} R_n \right) + \left(\frac{DU^\alpha}{D\theta^n} R_n \right) \Phi + U^\alpha \left(\frac{D\Phi}{D\theta^n} R_n \right) \right] \quad (14)$$

$$= \Phi^{-1} \left(\frac{DU^\alpha}{D\theta^n} R_n \right) \Phi, \quad (15)$$

if and only if

$$\text{pr } w_R (D_{x^\alpha} \Phi - U^\alpha \Phi) = 0, \quad \text{whenever} \quad D_{x^\alpha} \Phi - U^\alpha \Phi = 0. \quad (16)$$

To complete the proof given by Fokas-Gel'fand, we require the additional assumption that the vector field w_R also be a symmetry of the LSP $\Lambda([\Phi], \lambda) = 0$. However, it is nontrivial to identify which among the generalized symmetries w_R of the ZCC $\Delta[\theta] = 0$ are those which are common symmetries of the LSP $\Lambda([\Phi], \lambda) = 0$, *i.e.* whether condition (16) holds for all solutions

$\Phi \in G$ of the corresponding LSP (4), unless there exists a closed form of the wavefunction. To overcome these difficulties, we require that the two vector fields, w_R defined on the jet space N and

$$\eta_Q = Q^j [\Phi] \frac{\partial}{\partial \Phi^j}, \quad \Phi = \exp (\Phi^j ([\theta], \lambda) e_j) \quad (17)$$

(where $\{e_j\}$ is a basis in g) defined on the Lie group G be Φ -related (in the sense introduced in [10] page 33). This fact allows us to proceed with the construction of the symmetries common to both systems.

We now show that the symmetry associated with the vector field η_Q of the LSP (4) coincides with the symmetry w_R of the ZCC $\Delta [\theta] = 0$ if there exists, on the group G , a vector field

$$\sigma_R = R^k ([\theta]) \frac{\partial \Phi^j}{\partial \theta^k} \frac{\partial}{\partial \Phi^j} \Big|_{\Phi([\theta], \lambda)} \quad (18)$$

for which the relation

$$Q^j [\Phi ([\theta], \lambda)] = R^k ([\theta]) \frac{\partial \Phi^j}{\partial \theta^k} \Big|_{([\theta], \lambda)} \quad (19)$$

holds.

Indeed, for a smooth map Φ taking values in a neighborhood \mathcal{B} of the unit element $e \in G$, we use the canonical coordinates defined by the formula

$$\Phi : ([\theta], \lambda) \in (N \times \lambda) \rightarrow \Phi ([\theta], \lambda) = \exp(\Phi^j ([\theta], \lambda) e_j) \in G. \quad (20)$$

For other points which do not belong to \mathcal{B} one constructs a similar parametrisation in some neighborhood of a point $\Phi([\theta_0], \lambda)$. Using left translation, we have

$$\Phi ([\theta], \lambda) = \exp(\Phi^j ([\theta], \lambda) e_j) \Phi ([\theta_0], \lambda), \quad (21)$$

or equivalently

$$\exp(\Phi^j ([\theta], \lambda) e_j) = \Phi ([\theta], \lambda) \Phi^{-1} ([\theta_0], \lambda) \in G. \quad (22)$$

Assume that w_R is a symmetry of the ZCC $\Delta [\theta] = 0$ and η_Q is a symmetry of the LSP $\Lambda([\Phi], \lambda) = 0$. Now we relate these fields by means of a map Φ

$$d\Phi (w_R|_{[\theta]}) = d\Phi \left(R^k [\theta] \frac{\partial}{\partial \theta^k} \right) \Big|_{\Phi([\theta], \lambda)} = R^k [\theta] \frac{\partial \Phi^j}{\partial \theta^k} \frac{\partial}{\partial \Phi^j} \Big|_{\Phi([\theta], \lambda)} \quad (23)$$

and require that

$$d\Phi(w_R) = \eta_Q \quad (24)$$

in the sense that for every two points $[\theta], [\theta'] \in N$

$$\Phi([\theta], \lambda) = \Phi([\theta'], \lambda) \Rightarrow R^k[\theta] = R^k[\theta'] = Q^k[\Phi([\theta], \lambda)]. \quad (25)$$

In this case the vector fields w_R and η_Q are said to be Φ -related. In other words, we consider $d\Phi(w_R)$ as a vector field σ_R on a group G given by (18) for which the condition (19) holds. In this case we say that the ZCC $\Delta[\theta] = 0$ and the LSP $\Lambda([\Phi], \lambda) = 0$ have a common infinitesimal symmetry

$$\eta_Q = \sigma_R = d\Phi(w_R), \quad (26)$$

since the vector field σ_R is a symmetry of the LSP $\Lambda([\Phi], \lambda) = 0$ and the vector field w_R is a symmetry of the ZCC $\Delta[\theta] = 0$.

In conclusion, we have shown that, based on the invariance criterion for generalized symmetries, the necessary and sufficient conditions for the existence of a Fokas Gel'fand immersion function of a 2D-surface in a Lie algebra g requires that the generalized symmetry w_R of the ZCC $\Delta[\theta] = 0$ coincide with the symmetry η_Q of the LSP $\Lambda([\Phi], \lambda) = 0$. If the vector fields w_R and η_Q are Φ -related for which the condition (19) holds, then the 2D-soliton surface can be integrated explicitly and the obtained expression (9) is consistent with tangent vectors (2) whenever the matrices A^α are linearly independent. However, if the vector field w_R is a symmetry of the ZCC $\Delta[\theta] = 0$ but not a symmetry of the LSP $\Lambda([\Phi], \lambda) = 0$ (since the action of $\text{pr } w_R$ on the LSP does not vanish for all solutions Φ of the LSP), the g -valued immersion function F can still exist with linearly independent tangent vectors given by (2). The integrated expression for the immersion is not the Fokas-Gel'fand form given in (9), *i.e.* $F \neq \Phi^{-1} \text{pr } w_R \Phi$. This phenomenon has already been observed in several examples (see e.g. [7, 8, 9]). For instance, it was shown [7] that the surfaces induced by the conformal symmetry w_c of the completely integrable Euclidean $\mathbb{C}P^{N-1}$ sigma model induce a Fokas-Gel'fand immersion in the $\text{su}(N)$ Lie algebra as given by (9) and has the appropriate form of tangent vectors given by (2). However, it was shown [7] that for traveling wave solutions of the $\mathbb{C}P^{N-1}$ sigma model defined on Minkowski space, the Fokas-Gel'fand immersion functions (9) do not coincide in general with the immersion formula given by (1), since $A^\alpha([\theta, \lambda]) \neq \text{pr } w_c U^\alpha, \alpha = 1, 2$. So the

tangent vectors are not in the form given by (2) and the condition (19) does not hold. However the latter statements hold, for arbitrary conformal transformations, when one assumes the existence of finite action solutions for the $\mathbb{C}P^{N-1}$ sigma model defined on Euclidean space.

The method discussed in this communication is quite general and provides us with a procedure for constructing the explicit form of the Fokas-Gel'fand formula for immersion of 2D-soliton surfaces, associated with integrable systems, into Lie algebras. A classification of common symmetries of integrable systems and their linear spectral problems gives us the possibility of constructing such surfaces corresponding to any solution of the model. This will increase the range of applicability of this theory, including the classification of invariant geometrical characterization of surfaces and the nonlinear equations of the soliton theory.

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